WATER RESOURCES RESEARCH, VOL. 44, W03410, doi:10.1029/2007WR006135, 2008

New analytical derivation of the mean annual water-energy balance equation

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Received 24 April 2007; revised 6 December 2007; accepted 4 January 2008; published 12 March 2008.

[1] The coupled water-energy balance on long-term time and catchment scales can be expressed as a set of partial differential equations, and these are proven to have a general solution as $E/P = F(E_0/P, c)$, where c is a parameter. The state-space of (P, E_0, E) is a set of curved faces in $P - E_0 - E$ three-dimensional space, whose projection into $E/P - E_0/P$ two-dimensional space is a Budyko-type curve. The analytical solution to the partial differential equations has been obtained as $E = E_0P/(P^n + E_0^n)^{1/n}$ (parameter n representing catchment characteristics) using dimensional analysis and mathematic reasoning, which is different from that found in a previous study. This analytical solution is a useful theoretical tool to evaluate the effect of climate and land use changes on the hydrologic cycle. Mathematical comparisons between the two analytical equations showed that they were approximately equivalent, and their parameters had a perfectly significant linear correlation relationship, while the small difference may be a result of the assumption about derivatives in the previous study.

Citation: Yang, H., D. Yang, Z. Lei, and F. Sun (2008), New analytical derivation of the mean annual water-energy balance equation, *Water Resour. Res.*, *44*, W03410, doi:10.1029/2007WR006135.

1. Introduction

[2] The water-energy balance for a catchment over a long-term timescale describes the relationship between the components of water and heat balances of land [*Budyko*, 1974, p. 322], e.g., the partition of precipitation (P) into evapotranspiration (E) and runoff (R) controlled.

[3] Many attempts have been made to formulate the mean annual water-energy balance [*Schreiber*, 1904; *Ol'dekop*, 1911]. *Budyko* [1958] found that the actual evapotranspiration calculated by Schreiber' equation was lower than observed, while the values calculated by Ol'dekop's equation were higher than those observed, and hence he employed the geometric mean of the two equations. *Pike* [1964] suggested a different equation. *Budyko* [1974] summarized that the primary factors determining the rate of evapotranspiration for a long-term mean were the available energy and water. Since potential evapotranspiration (E_0) can measure the available energy, and precipitation (P) can represent the available water, the Budyko hypothesis can be expressed as

$$E/P = F_0(E_0/P), \tag{1}$$

in which the function F_0 was supposed to have a common form.

[4] In recent years, climate changes are increasingly significant [*Intergovernmental Panel on Climate Change* (*IPCC*), 2001]; the impact on the water cycle becomes a focus of hydrological and climatic studies. The Budyko hypothesis, as an important theoretical tool, has been widely used to appraise the impact. For evaluating the sensitivity of runoff (R) to precipitation (P), *Schaake* [1990] derived the climate elasticity

$$\varepsilon_P(P,R) = \frac{\partial R}{\partial P}\cdot\frac{P}{R}$$

($\partial R/\partial P$ was derived using Budyko curve). In order to forecast the change in runoff rate due to precipitation and potential evapotranspiration changes, hydrometeorologists [*Dooge et al.*, 1999; *Arora*, 2002] proposed a sensitivity factor for runoff,

$$\frac{\Delta R}{R} = \left[1 + \frac{\phi F_0'(\phi)}{1 - F_0(\phi)}\right]\frac{\Delta P}{P}\,\frac{\phi F_0'(\phi)}{1 - F_0(\phi)}\,\frac{\Delta E_0}{E_0}$$

($\phi = E_0/P$, and $F_0'(\phi)$ is the derivative with respect to ϕ). *Koster and Suarez* [1999] estimated the evaporation variability to be as climatic forcing as the evapotranspiration deviation ratio,

$$\frac{\sigma_E}{\sigma_P} = F_0(\phi) - \phi F_0'(\phi)$$

(σ_E and σ_P are the standard deviations of E and P, respectively). In these quantitative analyses, the derivatives of F_0 have been used.

[5] The form of F_0 (including these equations referred above) has no parameter, so it is unable to capture the role of landscape characteristics (including vegetation). Considering the effects of landscape characteristics, an adjustable parameter was introduced [*Choudhury*, 1999; *Zhang et al.*, 2001]. However, there is lack of hydrological consideration on choosing the particular form of water-energy balance equation. Table 1 summarizes the formulae commonly used

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0043-1397/08/2007WR006135$09.00

Table 1. Different Formulae for the Mean Annual Water-Energy Balance

Formula	Parameter	Reference
$E = P[1 - \exp(-E_0/P)]$	none	*Schreiber* [1904]
$E = E_0 \tanh(P/E_0)$	none	*Ol'dekop* [1911]
$E = P/[1 + (P/E_0)^2]^{0.5}$	none	*Pike* [1964]
$E = \{P[1 - \exp(-E_0/P)] \cdot E_0 \tanh(P/E_0)\}^{0.5}$	none	*Budyko* [1958]
$E = P/[1 + (P/E_0)^{\alpha}]^{1/\alpha}$	α	*Mezentsev* [1955] *Choudhury* [1999]
$E = P[1 + w(E_0/P)]/ \cdot [1 + w(E_0/P) + P/E_0]$	w	*Zhang et al.* [2001]

for representing the mean annual water-energy balance. In addition, vegetation also has an effect on hydrologic cycle, any changes of which should be captured in the mean annual water-energy balance equation. Therefore an analytical solution to the water-energy balance equation, which includes not only the catchment characteristics (including vegetation) but also their changes, is expected.

[6] *Bagrov* [1953] first attempted to derive the analytical equation for the mean annual water-energy balance by introducing a first derivative $dE/dP = 1 - (E/E_0)^n$. *Mezentsev* [1955] modified it as $dE/dP = [1 - (E/E_0)^n]^m$ by assuming $m = (n + 1)/n$. Integrating the equation above, he obtained

$$E = PE_0/\left(P^n + E_0^n\right)^{1/n}. \tag{2}$$

[7] However, it was not explained why the derivative of dE/dP was the form proposed by them. *Fu* [1981] described the Budyko hypothesis as partial differential equations $\partial E/\partial P = f(E_0 - E, P)$, when E_0 = const; and $\partial E/\partial E_0 = g(P - E, E_0)$, when P = const. Through dimensional analysis and mathematical reasoning, one analytical solution was derived [*Fu*, 1981; *Zhang et al.*, 2004] as

$$\frac{E}{P} = 1 + \frac{E_0}{P} - \left[1 + \left(\frac{E_0}{P}\right)^{\varpi}\right]^{1/\varpi} \text{ or}$$
$$\frac{E}{E_0} = 1 + \frac{P}{E_0} - \left[1 + \left(\frac{P}{E_0}\right)^{\varpi}\right]^{1/\varpi}. \tag{3}$$

[8] It can be recognized that the partial differential equations $\partial E/\partial P = f(E_0 - E, P)$ and $\partial E/\partial E_0 = g(P - E, E_0)$ describe the supposed conditions in which the derivative of E with respect to P (or E_0) can be given as a function of the variables $E_0 - E$ and P (or $P - E$ and E_0); under general conditions, it is nevertheless possible that the derivative of E with respect to P (or E_0) cannot be given in this form, but only as a function of the variables E_0, E and P instead. Therefore an analytical derivation of the water-energy balance equation, under general conditions, is required.

[9] This paper aims to prove the existence of a unique solution to the mean annual water-energy balance equation and to find the analytical solution under general conditions. This will supply a theoretical tool for further studies on the evapotranspiration of catchments with different landscapes, and the impacts of land use changes and climate changes on the water cycle. With the state space (P, E_0, E), we expect an increased understanding on the hydrological mechanisms implicated in the mean annual water-energy balance equation.

2. Theoretical Derivation of the Mean Annual Water-Energy Balance Equation

[10] For the long-term timescale, the soil moisture can reach an equilibrium state s_0 [*Eagleson*, 1978]; this equilibrium soil moisture can be expressed as a function of mean annual precipitation, potential and actual evapotranspiration:

$$s_0 = s(P, E_0, E). \tag{4}$$

[11] On the other hand, E can be given by the function of potential evapotranspiration (defined as the evapotranspiration capacity, and estimated using the Penman equation as recommended by *Shuttleworth* [1993], i.e., the apparent potential evapotranspiration referred by *Brutsaert and Parlange* [1998]), precipitation and soil moisture as $E = E(E_0, P, s_0)$. Mathematically, the mean annual actual evapotranspiration can be expressed as an implicit function as follows:

$$E = E[E_0, P, s(P, E_0, E)] = E(P, E_0, E). \tag{5}$$

[12] In equation (5), E is not expressed as a function of P and E_0, but instead as an implicit function of P, E_0, and E, since E indirectly represents the catchment surface characteristics (because E may be different in different catchments even if E_0 and P are the same). This also means that E depends on E_0 and P, as well as the catchment characteristics (including vegetation). When there is no water inflow from the adjacent catchments over the long-term mean, the zero-order boundary conditions for equation (5) can be given by

$$\begin{cases} E = E_0, P/E_0 \to \infty \\ E = 0, P = 0 \\ E = P, E_0/P \to \infty \\ E = 0, E_0 = 0 \end{cases}, \tag{6}$$

and the first-order boundary conditions can also be given by

$$\begin{cases} \dfrac{\partial E}{\partial P} = 0, \quad P/E_0 \to \infty, \text{ or } E = E_0 \\ \dfrac{\partial E}{\partial E_0} = 0, \quad E_0/P \to \infty, \text{ or } E = P \\ \dfrac{\partial E}{\partial P} = 1, \quad P \to 0, \ E_0 \neq 0 \\ \dfrac{\partial E}{\partial E_0} = 1, \quad E_0 \to 0, \ P \neq 0 \end{cases}. \tag{7}$$

[13] It is also possible to find out higher-order boundary conditions with further understanding of the evapotranspiration mechanism.

[14] On the basis of equation (5), the derivatives of E with respect to P, and E_0 can be expressed as follows, respectively:

$$\begin{cases} \dfrac{\partial E}{\partial P} = F(P, E_0, E) \\ \dfrac{\partial E}{\partial E_0} = G(P, E_0, E) \end{cases}, \tag{8}$$

in which $F(P, E_0, E)$ and $G(P, E_0, E)$ represent functions of P, E_0, and E. The total differential form of E can be given as

$$dE = \frac{\partial E}{\partial P} dP + \frac{\partial E}{\partial E_0} dE_0,$$

and substitution of equation (8) into this yields

$$F(P, E_0, E)dP + G(P, E_0, E)dE_0 - dE = 0, \tag{9}$$

which is a Pfaffian equation in mathematics.

[15] Mathematically, a necessary condition for equation (8) to have a solution is that equation

$$\frac{\partial F}{\partial E_0} + G\frac{\partial F}{\partial E} = \frac{\partial G}{\partial P} + F\frac{\partial G}{\partial E}$$

has to have one. This equation can transform into

$$G\left(-\frac{\partial F}{\partial E}\right) + F\left(\frac{\partial G}{\partial E}\right) + \left(\frac{\partial F}{\partial E_0} - \frac{\partial G}{\partial P}\right) = 0. \tag{10}$$

[16] For equation (10), it is essential that the Pfaffian equation be completely integrable. In other words, if equation (8) has a solution, equation (9) is completely integrable. Therefore an integrating factor exists as $\mu(P, E_0, E)$, and then multiplying both sides of equation (9) with this factor yields $dU = \mu(FdP + GdE_0 - dE) = 0$. Integrating this equation leads to following expression:

$$U(P, E_0, E) = c, \tag{11}$$

where c is a constant for a particular catchment. Equation (11) describes a set of curved faces with only one parameter in the state-space (P, E_0, E). In response to the theorem of the Pfaffian equation, if a Pfaffian equation is completely integrable in the domain D, any point in the domain D belongs to one and only one of these curved faces. It explains that the analytical solution to the mean annual water-energy balance equation must have a single parameter, and can be used for describing the domain D specified by the boundary conditions in equations (6) and (7). This also suggests the existence of a unique solution to the mean annual water-energy balance equation. One curved face describes the relationship of the water-energy balance for catchments with an identical parameter c which represents the catchment characteristics.

[17] According to the Buckingham pi theorem, we define two dimensionless variables as $\pi_1 = E_0/P$ and $\pi_2 = E/P$, and equation (11) transforms into $\pi_2 = F_1(\pi_1, c)$, i.e.,

$$E/P = F_1(E_0/P, c). \tag{12}$$

where F_1 represents a function. Equation (12) is similar to the Budyko hypothesis (equation (1)). In addition, this shows that the analytical equation of the mean annual water-energy balance has only one parameter.

[18] Only a single dimension is on the right-hand side of the equation (8), while a dimensionless number is on the left-hand side. We define two dimensionless variables as

$$x = \frac{P}{E}, y = \frac{E_0}{E}, \tag{13}$$

where x and y represent the effects of available water and energy on the evapotranspiration of catchments, respectively. According to the Buckingham pi theorem, equation (8) transforms into

$$\begin{cases} \dfrac{\partial E}{\partial P} = f(x,y) \\ \dfrac{\partial E}{\partial E_0} = g(x,y) \end{cases}. \tag{14}$$

[19] Assumption of P and E_0 being independent (i.e., $\partial P/\partial E_0 = 0$) yields the differentiation of equation (14):

$$\begin{cases} \dfrac{\partial^2 E}{\partial E_0 \partial P} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial E_0} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial E_0} = -\dfrac{P}{E^2} g \dfrac{\partial f}{\partial x} + \dfrac{E - E_0 g}{E^2}\dfrac{\partial f}{\partial y} \\ \dfrac{\partial^2 E}{\partial P \partial E_0} = \dfrac{\partial g}{\partial y}\dfrac{\partial y}{\partial P} + \dfrac{\partial g}{\partial x}\dfrac{\partial x}{\partial P} = -\dfrac{E_0}{E^2} f \dfrac{\partial g}{\partial y} + \dfrac{E - Pf}{E^2}\dfrac{\partial g}{\partial x} \end{cases}.$$

[20] As long as E is second-order continuous and differentiable, the equation

$$\frac{\partial^2 E}{\partial E_0 \partial P} = \frac{\partial^2 E}{\partial P \partial E_0}$$

can be obtained. Hence

$$-\frac{P}{E^2} g \frac{\partial f}{\partial x} + \frac{E - E_0 g}{E^2}\frac{\partial f}{\partial y} = -\frac{E_0}{E^2} f \frac{\partial g}{\partial y} + \frac{E - Pf}{E^2}\frac{\partial g}{\partial x}. \tag{15}$$

[21] To solve equation (15), two equations are given as

$$-\frac{P}{E^2} g \frac{\partial f}{\partial x} = -\frac{E_0}{E^2} f \frac{\partial g}{\partial y}, \tag{16a}$$

$$\frac{E - E_0 g}{E^2}\frac{\partial f}{\partial y} = \frac{E - Pf}{E^2}\frac{\partial g}{\partial x}. \tag{16b}$$

[22] The solution satisfying equations (16a) and (16b) is the solution to equation (15). One solution to equation (16a) is

$$\begin{cases} f(x,y) = x^{\alpha}\psi(y) \\ g(x,y) = y^{\alpha}\varphi(x) \end{cases}, \tag{17}$$

where $\varphi(x)$ is a function of x; $\psi(y)$ is a function of y. Substituting equations (13) and (17) into (16b) yields

$$y^{\alpha}\left[1 - x^{\alpha+1}\psi(y)\right]\varphi'(x) = x^{\alpha}\left[1 - y^{\alpha+1}\varphi(x)\right]\psi'(y). \tag{18}$$

[23] When $\alpha + 1 \neq 0$, it has

$$\begin{cases} \psi(y) = A_1 y^{\alpha+1} \\ \varphi(x) = A_1 x^{\alpha+1} \end{cases}, \tag{19}$$

where A_1 is an integral constant. When $\alpha + 1 > 0$, for the boundary condition $y \rightarrow \infty$ and $x \neq 0, f(x,y) = x^{\alpha} A_1 y^{\alpha+1} \rightarrow \infty$, i.e., $\partial E/\partial P \rightarrow \infty$, which does not satisfy the boundary condition $\partial E/\partial P = 1/P \rightarrow 0$, $E_0 \neq 0$. When $\alpha + 1 < 0$, for the boundary condition $x \rightarrow 1$, it has $y \rightarrow \infty$, so $f(x, y) =$

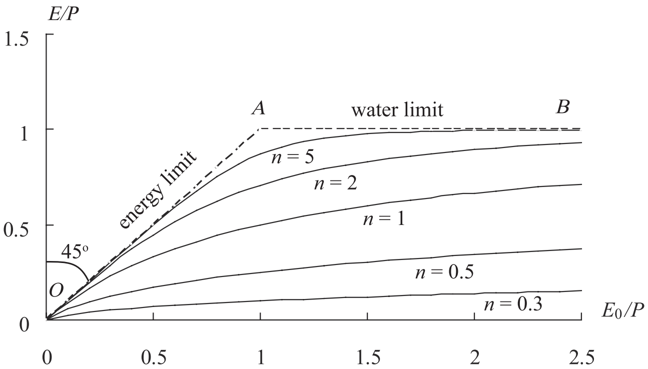

Figure 1. Solution space for the water-energy balance.

$x^\alpha A_1 y^{\alpha+1} \to 0$, and $y \neq 0$, i.e., $\partial E/\partial P \to 0$, which also does not satisfy the boundary condition $\partial E/\partial P = 1$, $P \to 0$, $E_0 \neq 0$. When $\alpha + 1 = 0$, equation (18) transforms into

$$\frac{x}{1-\varphi(x)}\frac{\partial\varphi(x)}{\partial x} = \frac{y}{1-\psi(y)}\frac{\partial\psi(y)}{\partial y}, \quad (20)$$

where x and y are independent variables, so both sides of equation (20) must equal a constant n, i.e.,

$$\frac{x}{1-\varphi(x)}\frac{\partial\varphi(x)}{\partial x} = \frac{y}{1-\psi(y)}\frac{\partial\psi(y)}{\partial y} = n.$$

[24] Integration leads to the following expression:

$$\begin{cases} \psi(y) = 1 + \dfrac{A}{y^n} \\ \varphi(x) = 1 + \dfrac{B}{x^n} \end{cases}, \quad (21)$$

where A and B are integral constants. Substituting equations (13), (14), and (17) into (21) yields

$$\begin{cases} \dfrac{\partial E}{\partial P} = \dfrac{E}{P}\left(1 + A\dfrac{E^n}{E_0^n}\right) \\ \dfrac{\partial E}{\partial E_0} = \dfrac{E}{E_0}\left(1 + B\dfrac{E^n}{P^n}\right) \end{cases}. \quad (22)$$

[25] Since $E = 0$ when $E_0 = 0$ or $P = 0$, therefore $E = PE_0\zeta(P, E_0)$ (where $\zeta(P, E_0)$ is a function of P and E_0), and substituting it into equation (22) yields

$$\begin{cases} A\zeta\dfrac{(PE_0\zeta)^n}{E_0^n} = P\dfrac{\partial\zeta}{\partial P} \\ B\zeta\dfrac{(PE_0\zeta)^n}{P^n} = E_0\dfrac{\partial\zeta}{\partial E} \end{cases}. \quad (23)$$

[26] Integration of equation (23) gives

$$\zeta(P, E_0) = \frac{1}{\left(-BP^n - AE_0^n + C\right)^{1/n}},$$

so

$$E = \frac{E_0 P}{\left(-BP^n - AE_0^n + C\right)^{1/n}}, \quad (24)$$

since $E = E_0$ when $P/E_0 \to \infty$, and therefore $B = -1$; $E = P$ when $E_0/P \to \infty$, and therefore $A = -1$; $E = P$ when $P \to 0$, and therefore $C = 0$. Therefore equation (24) becomes

$$E = \frac{E_0 P}{\left(P^n + E_0^n\right)^{1/n}}, \quad (25)$$

which has the same form as the equation proposed by *Mezentsev* [1955] and *Choudhury* [1999] but is different from the analytical equation given by *Fu* [1981].

3. Discussion

3.1. State Space of Mean Annual Water-Energy Balance

[27] The state space (P, E_0, E) is defined as the solution space of the mean annual water-energy balance equation, which is specified by two asymptotic faces. In the state space, one curved face, corresponding to a certain parameter c (or n), describes the water-energy balance for a certain catchment. In other words, when the parameter is given, the unique curved face is also determined (i.e., E is determined when P and E_0 are given for a certain catchment). As shown in Figure 1, two asymptotes join at point A at which E, P, and E_0 are equal; OA is the wet edge at which $E = E_0$, and AB is the dry edge at which $E = P$. The state space (P, E_0, E) is below the asymptote OAB. The state space (P, E_0, E) can be projected into the two-dimensional space (E_0/P, E/P), and the relationship between E_0/P and E/P is referred to as the *Budyko* [1974] hypothesis. In the two-dimensional space (E_0/P, E/P), the shape of the curve is determined by parameter n. The curve close to the x-axis ($n \to 0$) describes the water-energy balance relationship in the catchments with a very low water storage in the subsurface, such as the rocky, mountain catchments, where the precipitation completely transforms into runoff. The curve close to OAB ($n \to \infty$) describes the water-energy balance in the catchments with a very high water storage in the subsurface, such as the plain catchments with a deep quaternary soil layer, where E can reach the maximum (i.e., E_0 in a humid climate and P in an arid climate).

3.2. Comparing Different Formulae for the Mean Annual Water-Energy Balance

[28] In addition to equation (25) and Fu's equation, other equations, as listed in Table 1, can also be shown to satisfy the boundary conditions in equations (6) and (7). This indirectly confirms the validity of these boundary conditions. Nevertheless, their solution spaces are not equivalent to the state space (E_0/P, E/P), since, as an analytical solution, satisfying the boundary conditions is a necessary but not sufficient condition. In mathematics, a function $f_1(w)$ can be expressed as a series of $f_1(w) = \sum_{i=0}^{m} a_i(w - w_0)^i$, which represents the expansion at $w = w_0$. If the function form of $f_1(w)$ is unknown, to obtain its mathematical representation through a nonanalytical derivation may require boundary conditions from 0-order to m-order for fixing the parameter a_i (i = 0, 1, ..., m). Therefore it is possible that some equations, as given in Table 1, cannot satisfy the higher-order boundary conditions, but satisfies the 0-order and 1-order boundary conditions, and they only approximate the analytical solution as a result of neglecting higher-order differences. Different from a nonanalytical derivation, in an

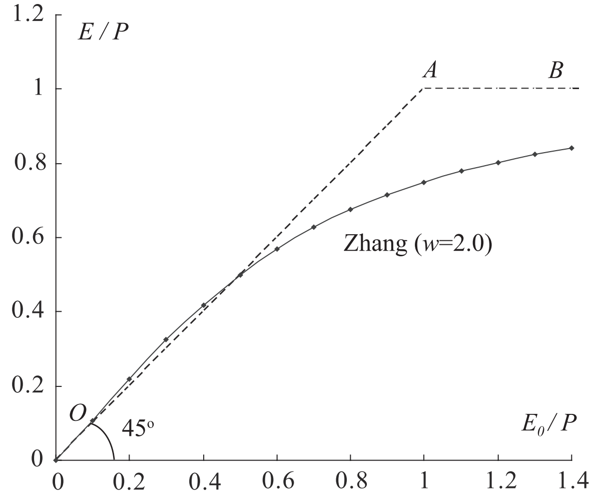

Figure 2. Solution to *Zhang et al.*'s [2001] equation with the parameter w = 2.0.

analytical derivation, boundary conditions are expected to resolve undetermined parameters (e.g., the integral constants when solving a partial differential equation).

[29] The solution space of the equation introduced by *Zhang et al.* [2001] is not equivalent to the state space (P, E_0, E), for example, point (0.40, 0.42) given by Zhang's equation ($w = 2$) is not included in the state space (Figure 2). This means that Zhang's equation does not agree with the wet boundary condition (the asymptote *OA*).

[30] It can be shown that the solution spaces given by Fu's equation (equation (3)) and equation (25) have the same asymptote *OAB* (see Appendix A), and they are the state space (P, E_0, E). It can be concluded that only Fu's equation and equation (25) are possible analytical solutions to the mean annual water energy balance equation.

3.3. Difference Between Fu's Equation and Equation (25)

[31] If there exists only a single analytical solution to the mean annual water-energy balance equation, Fu's equation and equation (25) should be comparable, and its necessary and sufficient condition is

$$P + E_0 - \left[P^{\varpi} + E_0^{\varpi}\right]^{1/\varpi} = \frac{E_0 P}{\left(P^n + E_0^n\right)^{1/n}}. \qquad (26)$$

[32] Dividing both sides by E_0, the above equation yields

$$(P/E_0) + 1 - \left[(P/E_0)^{\varpi} + 1\right]^{1/\varpi} = \frac{1}{\left[1 + (E_0/P)^n\right]^{1/n}}. \qquad (27)$$

[33] Define $z = P/E_0$, and we have

$$f_1(z) = 1 + z - \left[1 + z^{\varpi}\right]^{1/\varpi} - \frac{1}{\left[(1/z)^n + 1\right]^{1/n}} = 0. \qquad (28)$$

[34] If the two equations are comparable, parameter n should have a unique relationship with parameter ϖ, and the relationship should be independent from P, E, and E_0. The Taylor expansion of equation (28) at $z = 1$ can be written as

$$f_1(z) = \left(2 - 2^{1/\varpi} - 2^{-1/n}\right) + \frac{1}{2}\left(2 - 2^{1/\varpi} - 2^{-1/n}\right)(z-1) + \frac{1}{8}\left[(n+1)2^{-1/n} - (\varpi - 1)2^{1/\varpi}\right](z-1)^2 + O(z-1)^3 = 0. \qquad (29)$$

[35] In nonhumid regions, $z \leq 1$, neglecting the small quantity $(z-1)^2$ and its higher order, equation (29) transforms into

$$2^{1/\varpi} = 2 - 2^{-1/n}. \qquad (30)$$

[36] Similarly, in humid and subhumid regions, defining $z = E_0/P \leq 1$, and equation (28) will be obtained. Therefore equation (30) still comes into existence. Thus it can be concluded that the two equations are approximately similar solutions to the mean annual water-energy balance equation.

[37] In addition, the parameters ϖ and n were calibrated using long-term water balance data from 108 catchments of the nonhumid regions of China. These catchments have relatively few human alterations (e.g., dams and irrigation) to interfere with the water balance. Monthly discharge data for each catchment have been collected from 1960 to 2000. Daily meteorological data are available from 238 stations between 1960 and 2000. The procedures for calculating catchment average potential evapotranspiration (E_0) and precipitation (P) are (1) interpolating a 10-km grid data set covering the study areas from the gauge data; (2) estimating E_0 in each grid using the Penman equation recommended by *Shuttleworth* [1993]; and (3) calculating the catchment average E_0 and P. The actual evapotranspiration (E) was calculated on the basis of water balance (i.e., E equals precipitation minus runoff for mean annual) (see *Yang et al.* [2007] for more details). The high linear correlation between $2^{1/\varpi}$ and $2 - 2^{-1/n}$ is shown in Figure 3a. It proves

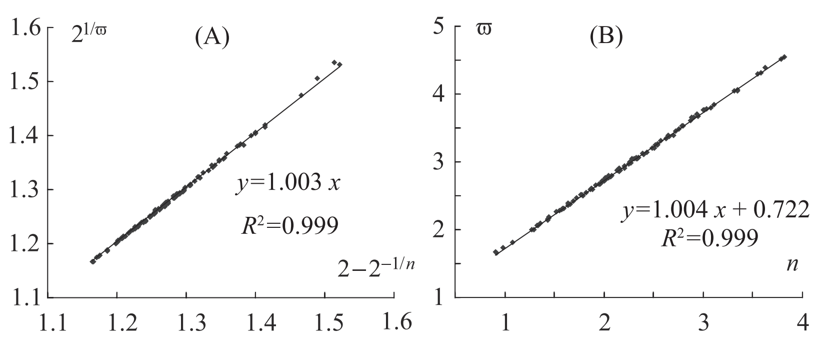

Figure 3. Relationship between the two parameters in the two equations (equation (25) in the present paper and *Fu*'s [1981] equation).

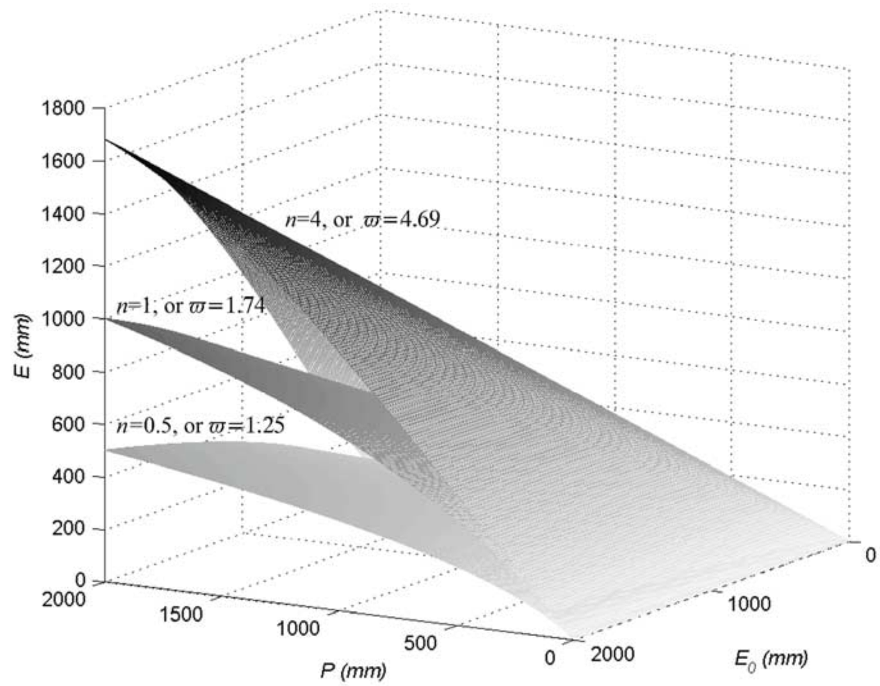

Figure 4. Solution to the mean annual water-energy balance equation in the three-dimensional state-space of $P - E_0 - E$.

equation (30) statistically. Additionally, the linear correlation of the two parameters is very high (R^2 = 0.999) as shown in Figure 3b; the linear regression equation is

$$\varpi = n + 0.72. \quad (31)$$

[38] Figure 4 illustrates the mean annual water-energy balance in a three-dimensional state space. It can be found that the three-dimensional curved faces given by the two equations are approximately equivalent when ϖ = 1.25, n = 0.5, ϖ = 1.74, n = 1, and ϖ = 4.69, n = 4. Figure 5 projects them into a two-dimensional state-space.

[39] Some subtle differences between the two equations should be notable. First, equation (30) is as a result of neglecting the small quantity $(z - 1)^2$ and its higher order in equation (29), and this will result in a small error. Second, some small differences can be observed in Figure 5, e.g., the curve with n = 1 agrees very well with Fu's result (ϖ = 1.74) when $E_0/P < 0.5$ or $E_0/P > 2$, while this is less than Fu's result when $0.5 < E_0/P < 2$. These differences may be the result from the assumptions made about the derivatives of E with respect to E_0 and P by *Fu* [1981] (i.e., $\partial E/\partial P = f(E_0 - E, P)$ when E_0 = const; and $\partial E/\partial E_0 = g(P - E, E_0)$ when P = const). In the present paper, we consider the partial differential equations as a general form with $\partial E/\partial P = f(E_0, P, E)$ and $\partial E/\partial E_0 = g(E_0, P, E)$ originating from the equation $E = E(P, E_0, E)$.

3.4. The Single Parameter of the Water-Energy Balance Equation

[40] Only a single parameter of the mean annual water-energy balance equation represents the integrated effects of catchment and vegetation characteristics, which has a significant effect on evapotranspiration. Factors affecting parameter n mainly include plant-available water holding capacity or root depth [*Milly*, 1994; *Wolock and McCabe*, 1999; *Laio et al.*, 2001; *Zhang et al.*, 2001; *Potter et al.*, 2005], average slope [*Zhang et al.*, 2004], vegetation type or land use [*Choudhury*, 1999; *Zhang et al.*, 2001; *Bounoua et al.*, 2004], vegetation cover [*Eagleson*, 2002; *Zhang et al.*, 2004], etc. The effect of every factor on the water-energy balance is mostly known. Nevertheless, because of strong cross correlations among these factors, to ascertain the key factors and derive an analytical equation for parameter n is not impossible, but difficult. Instead of an analytical equation, the empirical formula as proposed by *Yang et al.* [2007] correlates the parameter in Fu's equation with the relative infiltration capacity (the ratio of saturated hydraulic conductivity to mean precipitation intensity), relative soil water storage (the ratio of plant extractable

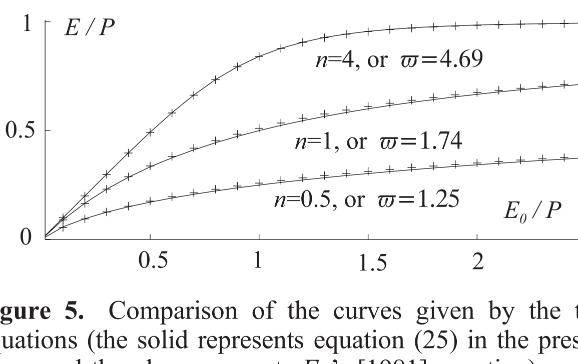

Figure 5. Comparison of the curves given by the two equations (the solid represents equation (25) in the present paper and the plus represents *Fu*'s [1981] equation).

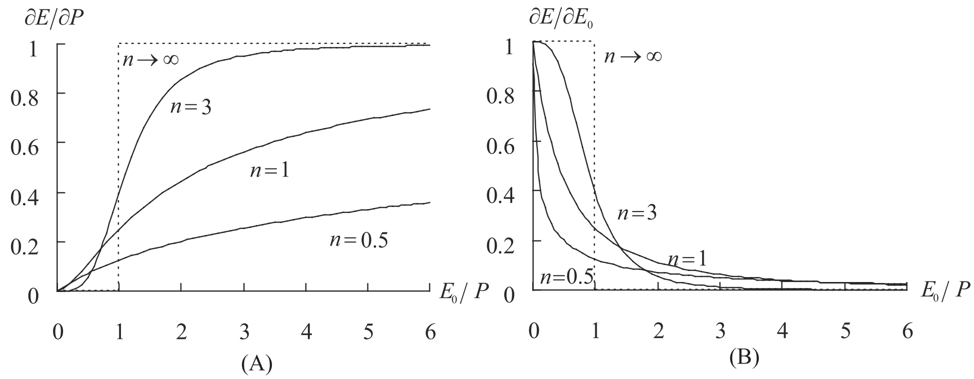

Figure 6. Relationship between (a) $\partial E/\partial P$ and E_0/P, and (b) $\partial E/\partial E_0$ and E_0/P calculated using equation (32).

water capacity to mean annual potential evapotranspiration), and the average slope. It is noted that this equation does not include the effect of vegetation except the plant extractable water capacity. Further studies are therefore required.

[41] Land use changes (e.g., deforestation, urbanization, farming, etc.) have an effect on the hydrologic cycle. As shown in Figure 5, when n increases from 0.5 to 1, E/P can increase by up to 50%. The effect can best be described with changes of n (dn), and after that the effect on evapotranspiration is calculated by $dE = \partial E/\partial n \cdot dn$.

3.5. Derivatives of This Equation

[42] As an analytical equation, equation (25) can be differentiated with respect to P or E_0. The derivatives describe the effect on the hydrologic cycle as a result of changing P or E_0 as follows:

$$\begin{cases} \partial E/\partial P = 1\Big/\left[1 + (P/E_0)^n\right]^{1+1/n} \\ \partial E/\partial E_0 = 1\Big/\left[1 + (E_0/P)^n\right]^{1+1/n} \end{cases}. \tag{32}$$

[43] In Figure 6a, if neglecting the changes in storage, the curves describe the partition of the increment of precipitation (dP) into the increments of evapotranspiration ($\partial E/\partial P \cdot dP$) and runoff ($dR = dP - \partial E/\partial P \cdot dP$). In humid regions ($E_0/P < 1$), most of dP transforms into runoff, while most evapotranspirates in arid regions ($E_0/P > 1$). Therefore runoff modifies more significantly when P changes in humid regions than in arid regions. In arid regions, the larger n becomes, the smaller the effect of dP on runoff is; however, in humid regions, when P increases, both an increase and a decrease of runoff is possible. Figure 6b illustrates the sensitive of E to E_0.

[44] As shown in Figure 7, the effect of climate changes on the hydrologic cycle can be described as a function of climatic characteristics ($\phi = E_0/P$) and catchments characteristics (n). When $\phi = 1$, it gives $\partial E/\partial P = \partial E/\partial E_0$; when $\phi < 1$, $\partial E/\partial P > \partial E/\partial E_0$, the change in evapotranspiration is dominated by the change in precipitation; when $\phi > 1$, the change in evapotranspiration is dominated by the change in potential evapotranspiration. And the larger n becomes, the larger the effect of climate changes on E is. According to n and ϕ, catchments can be classified, since the same ϕ and n determine the same sensitivity to changes in climate ($\partial E/\partial P$, $\partial E/\partial E_0$), as well as the same evapotranspiration ratio (E/P, the partition of precipitation). Subsequently, because $E + R = P$, we can obtain $\partial E/\partial P + \partial R/\partial P = 1$ and $\partial E/\partial E_0 + \partial R/\partial E_0 = 0$, where R represents runoff. Hence the same sensitivity of R to changes in climate ($\partial R/\partial P$, $\partial R/\partial E_0$) can be obtained. In addition, Figure 7 shows $\partial E/\partial P + \partial E/\partial E_0 \leq 1$.

3.6. Assumption About P and E_0 Being Independent

[45] In this paper, when deriving equation (25), it is assumed that P and E_0 are independent (i.e., $\partial P/\partial E_0 = 0$), as well as in *Fu's* [1981] derivation. Under a given climatic condition, we can consider P and E_0 as independent variables. In fact, because of the feedback of atmosphere on land surface, E increases as a result of increasing P; and E_0 decreases according to the complementary relationship between actual and potential evapotranspiration [*Bouchet*, 1963]. P, E_0, and E are therefore not independent, which also can be expressed as a point (P, E_0, E) in the state space or as $U(P, E_0, E) = c$. In other words, as a result of precipitation changing from P_1 to P_2, the state changes from an initial state (P_1, $E_{0,1}$, E_1) to a new state, not being (P_2, $E_{0,1}$, E_2), but (P_2, $E_{0,2}$, E_2). To estimate (P_2, $E_{0,2}$, E_2), two equations are set: one from the water-energy balance (considering P_2 and $E_{0,2}$ as given variables), and the other from the complementary relationship between potential and actual evapotranspiration [*Brutsaert and Stricker*, 1979;

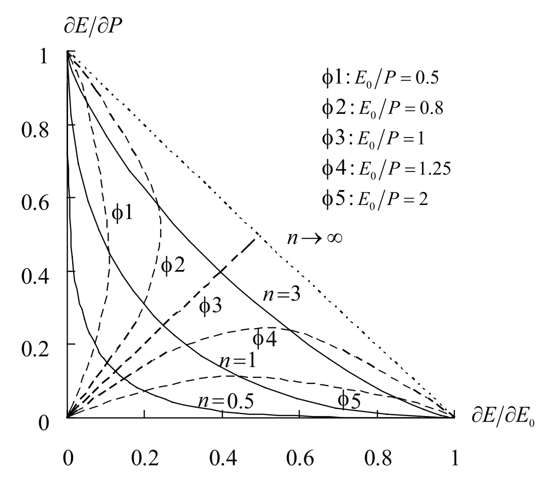

Figure 7. Relationship between $\partial E/\partial P$ and $\partial E/\partial E_0$ calculated using equation (32).

Parlange and Katul, 1992; *Brutsaert and Parlange*, 1998; *Kahler and Brutsaert*, 2006], i.e.,

$$bE + E_0 = (1 + b)E_{\rm w}. \tag{33}$$

where b is a constant of proportionality; $E_{\rm w}$ represents the wet environment evapotranspiration [*Brutsaert and Stricker*, 1979], which can be calculated from the net radiation ($R_{\rm n}$) by the Priestley-Taylor equation [*Priestley and Taylor*, 1972]. Additionally, these two equations have two independent variables (P and $R_{\rm n}$, not being interrelated, i.e., $\partial P/\partial R_{\rm n} = 0$), and therefore the two dependent variables (E and E_0) can be resolved. This implies that a relatively stabile state (P, E_0, E) can be reached under given catchments characteristics, radiation and precipitation. Simultaneously, equations $dE = \partial E/\partial P \cdot dP + \partial E/\partial E_0 \cdot dE_0$ (the differential of equation (25)) and $bdE + dE_0 = (1 + b)\ dE_{\rm w}$ (the differential of equation 33) lead to following expression:

$$dE = [\partial E/\partial P \cdot dp + (1 + b)\partial E/\partial E_0 \cdot dE_{\rm w}]/(1 + b\partial E/\partial E_0). \tag{34}$$

[46] Substituting $C_{\rm p}E_{\rm pa}$ (E_0 being proportional $C_{\rm p}$ to pan evaporation $E_{\rm pa}$) in equation (33) for E_0, *Brutsaert and Parlange* [1998] interpreted the evaporation paradox, increasing terrestrial evaporation (E) and decreasing pan evaporation. Further, *Brutsaert* [2006] evaluated the change in E on the basis of the change in $R_{\rm n}$, T, and $E_{\rm pa}$. According to equation (34), the response of the hydrologic cycle (not only dE but also $dR = dP - dE$) to climate changes $dE_{\rm w}$ (dT and $dR_{\rm n}$) and dP instead of $dE_{\rm pa}$, can be evaluated.

[47] As discussed above, the effect of vegetation on the hydrologic cycle is captured by parameter c (or n), but the effect of the hydrologic cycle on vegetation is not considered here. In fact, vegetation and hydrologic cycle are interactive. The climate and hydrologic cycle through water, energy, and carbon dioxide to the surface helps to determine the type and structure of the vegetation [*Eagleson*, 2002]. Therefore vegetation will change as a result of changes in climatological conditions, and it will reach a balanced state, possibly for a long time. The balance state can be expressed as (P, E_0, E, c) with c including the effect of vegetation. Thus the third equation relating c with other factors (maybe some empirical relation) is also expected. Then the state (P, E_0, E, c) can be obtained as the solution of the three equations, which is also determined by the parameters of the equations, climatic forcing (P and $R_{\rm n}$), etc. This indicates that the vegetation and hydrologic cycle will reach equilibrium under a given condition (e.g., land topography, radiation, precipitation). If any of these factors changes, the state will alter from (P_1, $E_{0,1}$, E_1, c_1) to (P_2, $E_{0,2}$, E_2, c_2), corresponding to the point moving from one curved face to another in the state space (P, E_0, E).

4. Conclusion

[48] Through dimensional analysis and mathematical reasoning, this paper mathematically derived the general solution to the mean annual water-energy balance equation, and proved its uniqueness. Additionally we obtained an analytical solution to the mean annual water-energy balance equation, which was different from *Fu*'s [1981] equation. Only a single parameter of the solution is able to capture the catchment characteristics (including vegetation) and their changes. Furthermore, the derivatives of this analytical solution supply a theoretical tool for the study on the effects of land use and climate changes on the water cycle.

[49] Mathematical comparison between equation (25) and Fu's equation shows that the two equations are very similar, although not exactly equivalent. Statistically, it was found that the two parameters in both equations have a significant linear relationship, and the two equations give the same solution space. There are nevertheless some subtle differences between the two equations, which is possibly because of the assumption about the derivatives in *Fu*'s [1981] derivation.

Appendix A: Asymptotes of Fu's Equation and Equation (25)

[50] Defining $x = E_0/P$, Fu's equation can be written by

$$g_1(x) = 1 + x - (1 + x^{\varpi})^{1/\varpi}, \tag{A1}$$

and equation (25) can be written by

$$g_2(x) = 1/[1 + (1/x)^n]^{1/n}. \tag{A2}$$

[51] First, we prove that OA (i.e., $y = x$) is the asymptote of equation $g_1(x)$, when $x \in (0, 1]$. The distance between the lines OA and (A1) can be given as follows:

$$x - g_1(x) = (1 + x^{\varpi})^{1/\varpi} - 1. \tag{A3}$$

[52] Equation (A3) is a monotone increasing function of x, so

$$(1 + x^{\varpi})^{1/\varpi} - 1 \le (1 + 1)^{1/\varpi} - 1 = 2^{1/\varpi} - 1.$$
$$\forall \varepsilon > 0, \exists \varpi_0 = \frac{\ln 2}{\ln(1 + \varepsilon)} + 1,$$

when

$$\varpi > \varpi_0, \varpi > \frac{\ln 2}{\ln(1 + \varepsilon)} + 1 > \frac{\ln 2}{\ln(1 + \varepsilon)},$$

i.e., $(1 + x^{\varpi})^{1/\varpi} - 1 < \varepsilon$.

[53] Second, we prove that AB (i.e., $y = 1$) is the asymptote of equation $g_1(x)$, when $x \in [1, +\infty)$. Similarly, the distance between the lines AB and (A1) can be expressed as

$$1 - g_1(x) = (1 + x^{\varpi})^{1/\varpi} - x. \tag{A4}$$

[54] Equation (A4) is a monotone decreasing function of x, so

$$(1 + x^{\varpi})^{1/\varpi} - x \le (1 + 1)^{1/\varpi} - 1 = 2^{1/\varpi} - 1$$
$$\forall \varepsilon > 0, \exists \varpi_0 = \frac{\ln 2}{\ln(1 + \varepsilon)} + 1,$$

when

$$\varpi > \varpi_0, \varpi > \frac{\ln 2}{\ln(1 + \varepsilon)} + 1 > \frac{\ln 2}{\ln(1 + \varepsilon)},$$

i.e., $(1 + x^{\varpi})^{1/\varpi} - x < \varepsilon$. Consequently, Fu's equation has the asymptote OAB.

[55] In the same way, for equation (A2), when $x \in (0, 1]$, the difference between the lines OA and (A2) can be expressed as

$$x - g_2(x) = x - 1/[1 + (1/x)^n]^{1/n}. \quad \text{(A5)}$$

[56] The derivative of equation (A5) is

$$1 - \frac{1}{(x^n + 1)^{1/n}(x^n + 1)} > 0,$$

so equation (A5) is a monotone increasing function of x. Therefore

$$x - 1/[1 + (1/x)^n]^{1/n} \leq 1 - 2^{-1/n},$$
$$\forall \varepsilon > 0,\ \exists N = -\frac{\ln(1 - \varepsilon)}{\ln 2} + 1,$$

when

$$n > N, n > -\frac{\ln(1 - \varepsilon)}{\ln 2} + 1 > \frac{\ln(1 - \varepsilon)}{\ln 2},$$

i.e., $x - 1/[1 + (1/x)^n]^{1/n} < \varepsilon$.

[57] When $x \in [1, +\infty)$, we can express the distance between the lines AB and (A2) as follows:

$$1 - g_2(x) = 1 - 1/[1 + (1/x)^n]^{1/n}. \quad \text{(A6)}$$

[58] Equation (A4) is a monotone decreasing function of x, so

$$1 - 1/[1 + (1/x)^n]^{1/n} \leq 1 - 2^{-1/n},$$
$$\forall \varepsilon > 0, \exists N = -\frac{\ln(1 - \varepsilon)}{\ln 2} + 1,$$

when

$$n > N, n > -\frac{\ln(1 - \varepsilon)}{\ln 2} + 1 > -\frac{\ln(1 - \varepsilon)}{\ln 2},$$

i.e.,

$$1 - 1/[1 + (1/x)^n]^{1/n} < \varepsilon.$$

[59] Therefore equation (25) also has the asymptote OAB.

[60] **Acknowledgments.** This research was supported by the National Natural Science Foundation of China (50679029) and by the National 973 Project of China (2006CB403405). The authors would like to express their appreciation to Marc Parlange, Michiaki Sugita, Murugesu Sivapalan, and three anonymous reviewers, whose comments and suggestions led to significant improvements in the manuscript.

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